

ON CHARACTERIZING THE STANDARD QUANTUM LOGICS

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ABSTRACT. Let \mathcal{L} be a complete projective logic. Then \mathcal{L} has a natural representation as the lattice of $\langle \cdot, \cdot \rangle$ -closed subspaces of a left vector space V over a division ring D , where $\langle \cdot, \cdot \rangle$ is a definite θ -bilinear symmetric form on V , θ being some involutive antiautomorphism of D . Now a well-known theorem of Piron states that if D is isomorphic to the real field, the complex field or the sfield of quaternions, if θ is continuous, and if the dimension of \mathcal{L} is properly restricted, then \mathcal{L} is just one of the standard Hilbert space logics.

Here we also assume \mathcal{L} is a complete projective logic. Then if every θ -fixed element of D is in the center of D and can be written as $\pm d\theta(d)$, some $d \in D$, and if the dimension of \mathcal{L} is properly restricted, we show that \mathcal{L} is just one of the standard Hilbert space logics over the reals, the complexes, or the quaternions. One consequence is the extension of Piron's theorem to discontinuous θ . Another is a purely lattice theoretic characterization of the lattice of closed subspaces of separable complex Hilbert space.

1. Introduction. A point of particular interest in the foundations of quantum theory is the apparently unique role of the logics based on classical Hilbert space. Efforts to account for this have focused largely on finding simple or physically plausible assumptions concerning a logic which will lead naturally to a characterization of the standard Hilbert space derived logics [18], [19], [14], [7], [8], [11], [12], [4]. This may be viewed as a two stage process. First, as suggested by the original work of Birkhoff and von Neumann [3], there are hypotheses relating the structure of the logic to the finite dimensional projective geometries. From this the coordinatization of the logic is deduced, i.e., the logic is projective. From this point assumptions must be found which relate the coordinatizing division ring D to the field of real numbers. Zierler [18] assumes the compactness of certain sublattices of the

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logic together with a continuous nonconstant function from the unit interval to some finite dimensional sublattice of the logic, while Piron [14] simply postulates a rather direct relation between R and D . More recently, Gudder and Piron [7], by assuming the existence of a smooth maximal observable, and Maczynski [12], by making assumptions on states and an observable, are able to deduce that D is an extension of the real field R . If D is a finite extension it follows that D is either the real field, the complex field or the sfield of quaternions. We are principally concerned with this second stage. Our approach involves certain weak assumptions on D and θ , as set forth above, which must be satisfied if the elements of V are to give rise to states on the logic in a natural manner as occurs with the standard logics. These hypotheses are discussed in §3 while important implications in the general and infinite dimensional cases are derived in §4 and §5, respectively. As an application, in §6 we present a system of axioms characterizing the lattice of closed subspaces of separable complex Hilbert space using only concepts common to lattice theory.

2. Preliminaries. Though we shall in general refer the reader to Varadarajan [15] for terminology and standard results concerning logics, projective geometries and orthocomplemented vector lattices, it seems appropriate to give here the definition of a projective logic, which is the central theme of our discussion. As given in [15] a logic \mathcal{L} is *projective* if and only if it satisfies:

- (i) given $a \neq 0$ in \mathcal{L} , there is a point $x < a$;
- (ii) if $a \neq 0$ in \mathcal{L} is the (lattice) sum of a finite set of points, then $\mathcal{L}[0, a]$ is a geometry of finite rank;
- (iii) if $x, a \in \mathcal{L}$, $a \neq 0, \neq 1$ and x is a point, then there are points y and $z \in \mathcal{L}$ such that $y < a, z < a^\perp$ and $x < y \vee z$;
- (iv) there exists at least one $a \in \mathcal{L}$ such that $4 \leq \dim \mathcal{L}[0, a] < \infty$.

The key to our discussion is the fact that complete projective logics have an equivalent formulation in terms of certain orthocomplemented vector lattices. Let D be a division ring; V a left vector space over D with $4 \leq \dim V$; θ an involutive antiautomorphism of D ; and $\langle \cdot, \cdot \rangle$ a definite, symmetric ($\theta(\langle x, y \rangle) = \langle y, x \rangle$), θ -bilinear ($\langle ax, by \rangle = a\langle x, y \rangle\theta(b)$ and distributive) form on $V \times V$. If $M \subseteq V$ set

$$M^\perp = \{x \in V: \langle x, y \rangle = 0, \text{ for all } y \in M\}.$$

Then a subspace M of V is called closed if $M^{\perp\perp} = M$; and the set $\mathcal{L}(V, \langle \cdot, \cdot \rangle)$ of all closed subspaces of V under the natural inclusion forms a complete lattice with $^\perp$ as an orthocomplementation. If, in addition, for each $M \in \mathcal{L}(V, \langle \cdot, \cdot \rangle)$, $V = M + M^\perp$ algebraically, then $(V, \langle \cdot, \cdot \rangle)$ is said to be Hilbertian, and in this case $\mathcal{L}(V, \langle \cdot, \cdot \rangle)$ is a complete projective logic. In fact all complete projective logics are represented by such $\mathcal{L}(V, \langle \cdot, \cdot \rangle)$ with

$(V, \langle \cdot, \cdot \rangle)$ Hilbertian. For the proof see [15, p. 179]. If $(V, \langle \cdot, \cdot \rangle)$ is Hilbertian we shall refer to $(V, \langle \cdot, \cdot \rangle, \theta, D)$ as a *Hilbertian space*. Thus there is an equivalence between the complete projective logics and the Hilbertian spaces.

Finally it is important to note that if $(V, \langle \cdot, \cdot \rangle, \theta, D)$ and $(V', \langle \cdot, \cdot \rangle, \theta', D')$ are two Hilbertian spaces which yield the same complete projective logic then D and D' are isomorphic in such a way that θ and θ' correspond naturally to each other while $\langle \cdot, \cdot \rangle$ carries over to the primed space to a multiple of $\langle \cdot, \cdot \rangle'$ [15, pp. 33, 59].

3. The probabilistic hypotheses. A celebrated theorem of Gleason [6] asserts that all of the states on a standard logic arise naturally from normalized elements of any representing Hilbert space. While it is unknown whether this result also holds for the class of complete projective logics, yet if $(V, \langle \cdot, \cdot \rangle, \theta, D)$ is a Hilbertian space the elements of V deserve consideration as a possible source of states on the accompanying logic. With this in mind we shall consider some conditions which a Hilbertian space must satisfy if each nonzero vector is to yield a state in the usual manner.

Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be a Hilbertian space and let $x \in V$ with $x \neq 0$. Then foremost among our requirements would be the existence of a $d \in D$ with $\langle dx, dx \rangle = \pm 1$, i.e., x must be normalizable. Clearly $\langle dx, dx \rangle = \pm 1$ if and only if $\langle x, x \rangle = \pm d^{-1}\theta(d^{-1})$. Thus we might rephrase the normalizability condition as

(†) for each $x \in V$ there is a $d \in D$ with $\langle x, x \rangle = \pm d\theta(d)$.

Now while the condition (†) is equivalent to the existence of normalizations for a given Hilbertian space, it is not entirely satisfactory because it can happen that one Hilbertian space may satisfy it while another Hilbertian space representing the same complete projective logic may not. An example may be constructed as follows. Let A be a subfield of the real numbers in which $\sqrt{2}$ is never a sum of squares. Let s be a sum of squares of elements in A . We claim $\sqrt{2}$ is again not a sum of squares in $A(\sqrt{s})$. If $\sqrt{s} \in A$, this is obvious; so suppose $\sqrt{s} \notin A$. Then each element of $A(\sqrt{s})$ has the form $a + b\sqrt{s}$ with $a, b \in A$. Suppose

$$\sqrt{2} = \sum_{i=1}^n (a_i + b_i\sqrt{s})^2.$$

Then rearranging terms yields

$$\sqrt{2} = \left(\sum_{i=1}^n a_i^2 + \left(\sum_{i=1}^n b_i^2 \right) s \right) + \left(\sum_{i=1}^n 2a_i b_i \right) \sqrt{s}.$$

Clearly the coefficient of \sqrt{s} in this equation must be nonzero or $\sqrt{2}$ would

be a sum of squares in A . But then squaring both sides of the equation shows that $\sqrt{s} \in A$, a contradiction. Now starting with the field of rational numbers and repeatedly adjoining square roots of sums of squares we will arrive, after a transfinite process, at a field A which has a square root for each sum of squares but cannot have a square root for $\sqrt{2}$ for that would make $\sqrt{2}$ a sum of squares. Let V^n be a vector space of dimension $n > 4$ over A and let $\{e_i\}_{i=1}^n$ be a basis for V^n . Define $\langle \cdot, \cdot \rangle$ by $\langle e_i, e_j \rangle = \delta_{ij}$ and let θ be the identity isomorphism on A . Then $(V^n, \langle \cdot, \cdot \rangle, \theta, A)$ is a Hilbertian space which satisfies (\dagger) . However $(V^n, \langle \cdot, \cdot \rangle, \sqrt{2}, \theta, A)$ is another Hilbertian space which yields the same complete projective logic but in which no element is normalizable. Thus in order that the existence of normalizations be an invariant of the underlying logic, and to avoid what seems a pathological situation, a slightly stronger assumption than (\dagger) is necessary. Returning to the general Hilbertian space $(V, \langle \cdot, \cdot \rangle, \theta, D)$, let $F = \{d \in D: \theta(d) = d\}$. Then though the details are elementary and best omitted, the weakest assumption which guarantees the condition (\dagger) for all Hilbertian spaces with the same underlying logic as $(V, \langle \cdot, \cdot \rangle, \theta, D)$ is

(P₁) for each $f \in F$ there is a $d \in F$ with $f = \pm d\theta(d)$.

If C represents the center of D we shall also impose on $(V, \langle \cdot, \cdot \rangle, \theta, D)$ the additional condition

(P₂) $F \subseteq C$.

This condition is likewise independent of the representing Hilbertian space, i.e., an invariant of the logic. Its justification is that if $\langle x, x \rangle$ is to be a candidate for a probability then at least it should behave as a scalar and multiply commutatively in D . (Note that as one consequence of P₁, for each $f \in F$ there is an $x \in V$ with $f = \pm \langle x, x \rangle$.) We shall call a complete projective logic *probabilistic* if and only if some representing Hilbertian space satisfies P₁ and P₂.

A condition which is sometimes easier or more convenient to apply than P₁ and P₂ is

(N) given any $f \in F$, one of f or $-f$ has a square root in C .

This is another invariant of the logic and is somewhat stronger than P₁ and P₂. Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be a Hilbertian space satisfying condition N. Then clearly P₂ holds. Now let $f \in F$ and suppose for $c \in C$, $f = \pm c^2$. Then $f = \theta(f) = \pm (\theta(c))^2$ and since $\theta(c)$ is also an element of the field C we must have $\theta(c) = \pm c$. It follows that $f = \pm c\theta(c)$ and P₁ is established. We conclude that any complete projective logic represented by a Hilbertian space satisfying condition N is probabilistic.

4. General results. Here we shall assume $(V, \langle \cdot, \cdot \rangle, \theta, D)$ is a Hilbertian space satisfying conditions P_1 and P_2 and shall present some results concerning the structure of F and D .

First a remark on orientation. If for some $x \in V$, $\langle x, x \rangle = 1$ we shall say $(V, \langle \cdot, \cdot \rangle, \theta, D)$ is *positively oriented* while if $\langle x, x \rangle = -1$ it will be called *negatively oriented*. To see that these categories are well defined, suppose $x, y \in V$ with $\langle x, x \rangle = 1$ and $\langle y, y \rangle = -1$. Then if M is the subspace of V spanned by x and y , $M \in \mathcal{L}(V, \langle \cdot, \cdot \rangle)$ and since $V = M + M^\perp$ and $\dim V \geq 4$, $M^\perp \neq \{0\}$. Let $z \in M^\perp$ and suppose $\langle z, z \rangle = \pm 1$. If $\langle z, z \rangle = 1$ then $\langle y + z, y + z \rangle = 0$ while if $\langle z, z \rangle = -1$ then $\langle x + z, x + z \rangle = 0$. Either possibility contradicts the definiteness of $\langle \cdot, \cdot \rangle$. Throughout the remainder of our discussion we shall assume the positive orientation when convenient and only mention orientation when we feel it merits such attention.

4.1. THEOREM. *The field F has a natural ordering in which the set of positive elements is $P = \{\langle x, x \rangle : x \in V, x \neq 0\}$ if the space is positively oriented and is the negative of this set if the space is negatively oriented.*

PROOF. Since the positive and negative orientations differ by only a change in the sign of $\langle \cdot, \cdot \rangle$ it is sufficient to consider the positive orientation. Clearly $0 \notin P$. Let $f \in F$ and let $x \in V$ with $\langle x, x \rangle = 1$. Then there is a $d \in D$ with $f = \pm d\theta(d)$, and it follows that $f = \pm \langle dx, dx \rangle$. We have shown that $P \cup \{0\} \cup -P = F$. Next let $a, b \in P$ with $a = \langle x, x \rangle$ for some $x \in V$. By P_1 there is a $d \in D$ with $b = d\theta(d)$ (positive orientation), and by P_2 ,

$$ab = \langle x, x \rangle d\theta(d) = d\langle x, x \rangle \theta(d) = \langle dx, dx \rangle \in P.$$

We have $P \cdot P \subseteq P$. Now let $z \in V$ with $\langle x, z \rangle = 0$ and $\langle z, z \rangle = 1$. Then

$$a + b = \langle x + dz, x + dz \rangle \in P$$

and $P + P \subseteq P$, and since $0 \notin P$ also, $P \cap -P = \emptyset$. This concludes the proof.

This result has some implications of interest. Eckman and Zabey [5] have shown that quantum mechanics is impossible (there are no orthocomplementations) in a Hilbert space over a finite or Galois field unless the Hilbert space is of dimension 2 and the characteristic of the field is not 2. Their method is to enumerate the subspaces of various dimensions. Likewise Beltrametti and Cassinelli [1] have shown that quantum mechanics is impossible over any finite extension of a p -adic number field provided the dimension of the linear space is greater than or equal to five and the involutive antiautomorphism is assumed to be continuous. Here 4.1 shows that quantum mechanics is impossible in a space of dimension at least 4 (3 is actually sufficient for the argument, see [15, p. 61]) over any field of nonzero

characteristic provided there are enough square roots to satisfy the hypothesis N. In particular, all algebraically closed fields of nonzero characteristic are included. That some such hypothesis concerning the algebraic nature of the field is necessary may be seen from the following example. Let A be an arbitrary field and let $n \geq 4$ be a positive integer. Let $\{x_1, x_2, x_3, \dots, x_n\}$ be a set of n distinct indeterminants over A and set $B = A(x_1, x_2, x_3, \dots, x_n)$. Let U be a vector space over B of dimension n with basis $\{e_i\}_{i=1}^n$. Then with $\theta = i_B$ and $\langle \cdot, \cdot \rangle$ defined by

$$\left\langle \sum_{i=1}^n a_i e_i, \sum_{j=1}^n b_j e_j \right\rangle = \sum_{i=1}^n a_i b_i x_i$$

on $U \times U$, $(U, \langle \cdot, \cdot \rangle, \theta, B)$ is a Hilbertian space.

4.2. THEOREM. *The division ring D has one of the following three forms:*

- (i) $D = F$;
- (ii) $D = F(I)$ where I^2 is a negative element of F in the natural order;
- (iii) $D = F(I, J)$ where $IJ = -JI$ and I^2 and J^2 are negative elements of F in the natural order.

In case (ii) $\theta(I) = -I$ and in case (iii) $\theta(I) = -I$ and $\theta(J) = -J$.

PROOF. Suppose $D \neq F$ and let d be an element not in F . Then $d + \theta(d) = f \in F$ and, since F has characteristic zero, $f/2 \in F$. Let $I = d - (f/2)$. Then $\theta(I) = -I$ and hence $\theta(I^2) = I^2 \in F$. Let $x \in V$ with $\langle x, x \rangle = 1$. Then $\langle Ix, Ix \rangle = -I^2$ so I^2 is negative in the natural order of F . If $D = F(I)$ we have case (ii). Suppose $D \neq F(I)$. Let e be any element not in $F(I)$. Then $e + \theta(e) = g \in F$ and, setting $J' = e - (g/2)$, we have, analogous to the result for I , $\theta(J') = -J'$. Let $IJ' + \theta(IJ') = a \in F$. Then $\theta(IJ') = -IJ' + a = J'I$. Set $J = J' - (a/2I^2)I$. Then

$$JI = J'I - (a/2) = -IJ' + (a/2) = -IJ.$$

Since $\theta(J) = -J$ it follows as for I that J^2 is negative in the natural order of F . If $D = F(I, J)$ we have case (iii). It remains to show D cannot be larger than $F(I, J)$. Let r be an element not in $F(I, J)$. Then $r + \theta(r) = h \in F$, and setting $P = r - (h/2)$ we have $\theta(P) = -P$ and $P \notin F(I, J)$. Now set $IP + \theta(IP) = b \in F$ and $JP + \theta(JP) = c \in F$ and let $Q = P - (b/2I^2)I - (c/2J^2)J$. Then a simple calculation shows that $IQ = -QI$ and $JQ = -QJ$. Again $Q \notin F(I, J)$. Now, however, $(IJ)Q = Q(IJ)$, and since

$$\theta((IJ)Q) = \theta(Q)\theta(IJ) = (-Q)(-IJ) = Q(IJ),$$

we have $(IJ)Q \in F$ and hence $Q \in F(I, J)$. This contradiction completes the proof.

5. Infinite dimensional spaces and logics. The results of the previous section show that a complete projective logic which is probabilistic bears a close

relation structurally to one of the classical logics over the reals, complexes, or quaternions. Here we shall investigate the consequences of the additional assumption that the logic is infinite dimensional.

In the case that $(V, \langle \cdot, \cdot \rangle, \theta, D)$ represents an infinite dimensional logic, $(V, \langle \cdot, \cdot \rangle, \theta, D)$ will be referred to as an *infinite dimensional* Hilbertian space. This is equivalent to the statement that V is an infinite dimensional vector space over D .

Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be an infinite dimensional Hilbertian space. Then it is possible, and convenient for our purposes, to give meaning to the notion of infinite sum. Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence from V (i.e., $\langle u_n, u_m \rangle = \delta_{nm}$). Let M be the subspace of V spanned by $\{u_n\}_{n=1}^{\infty}$. Then if $x \in M^{\perp\perp}$ and $\langle x, u_n \rangle = b_n$ for $n = 1, 2, \dots$, we shall write

$$x = \sum_{n=1}^{\infty} b_n u_n.$$

This gives a unique representation for the elements of $M^{\perp\perp}$ which agrees with the usual sum in the case when only finitely many of the b_n 's are nonzero.

Before proceeding with our analysis of the infinite dimensional Hilbertian spaces we shall need a lemma concerning separable Hilbert space \mathcal{H} over the real numbers \mathbf{R} . Elements of \mathcal{H} will be written as infinite sequences (λ_i) and (\cdot, \cdot) will denote the inner product in \mathcal{H} . If A is any subfield of \mathbf{R} define $\mathcal{H}(A)$ by

$$\mathcal{H}(A) = \{(\lambda_i) \in \mathcal{H} \mid \lambda_i \in A \text{ for all } i\}.$$

Let \mathcal{H}° denote the finitely nonzero elements of \mathcal{H} .

5.1. LEMMA. *Let A be a subfield of \mathbf{R} and let $x_0 = (\lambda_i)$ and $y_0 = (\mu_i)$ be orthogonal elements of $\mathcal{H}(A)$. Then there are sequences $\{x_i\}$ and $\{y_i\}$ such that*

- (i) $x_i, y_i \in \mathcal{H}^\circ \cap \mathcal{H}(A)$ for all $i, j \geq 1$;
- (ii) $x_i \perp y_j$ for all $i, j \geq 0$;
- (iii) $x_i \rightarrow x_0$ and $y_i \rightarrow y_0$ in \mathcal{H} .

PROOF. Let $x_1 = y_1 = 0$. Then we need only consider the recursion step. Thus we assume $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ have all been defined. Let $\{y_{i(p)}\}_{p=1}^m$ denote a maximal independent subsequence of $\{y_i\}_{i=1}^n$. Then set

$$y'_{i(p)} = y_{i(p)} - \sum_{q=1}^{p-1} (y_{i(p)}, y'_{i(q)}) (y'_{i(q)}, y'_{i(q)})^{-1} y'_{i(q)} \quad \text{for } p = 1, 2, 3, \dots, m$$

to define recursively the orthogonal set $\{y'_{i(p)}\}_{p=1}^m$. Let

$$z = y_0 - \sum_{p=1}^m (y_0, y'_{i(p)}) (y'_{i(p)}, y'_{i(p)})^{-1} y'_{i(p)} = (v_i).$$

Now choose r large enough so that all y_i , $i < n$, are zero beyond the r th place and

$$\left| \sum_{i=1}^r \lambda_i v_i \right| < 2^{-(n+1)} \left| \sum_{i=1}^r v_i^2 \right|$$

and

$$\left| \sum_{i=r}^{\infty} \lambda_i^2 \right| < 2^{-(n+1)}.$$

Let $x' = (\lambda_1, \dots, \lambda_r, 0, 0, 0, \dots)$ and $z' = (v_1, \dots, v_r, 0, 0, 0, \dots)$. Then set

$$\begin{aligned} x_{n+1} &= x' \quad \text{if } z' = 0, \\ &= x' - (x', z')(z', z')^{-1}z' \quad \text{otherwise.} \end{aligned}$$

Clearly x_{n+1} has suitable coefficients and the right orthogonality properties. Also

$$\|x - x_{n+1}\| \leq \|x - x'\| + \|(x', z')(z', z')^{-1}z'\| \leq 2^{-(n+1)} + 2^{-(n+1)}\|y_0\|.$$

Now repeat the procedure beginning with $\{x_i\}_{i=1}^{n+1}$ and $\{y_i\}_{i=1}^n$ to obtain a y_{n+1} . This gives $\{x_i\}_{i=1}^{n+1}$ and $\{y_i\}_{i=1}^{n+1}$ with the desired properties and completes the recursion. Thus we have defined $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ which satisfy conditions (i)–(iii).

We shall now proceed to establish a kind of completeness property for infinite dimensional Hilbertian spaces satisfying P_1 and P_2 . Actually a hypothesis somewhat weaker than P_1 and P_2 will suffice. We shall require:

(OR) The fixed set F is an ordered subfield of C and all the $\langle x, x \rangle$, $x \in V$, are nonnegative in F .

We shall use the term absolute value with its standard meaning in an ordered field. In accordance with the usage for non-Archimedean fields, an element of F whose absolute value is less than any positive rational will be called infinitesimal while an element whose absolute value is greater than any rational will be termed infinite.

5.2. LEMMA. *Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ satisfy OR. Then for any $x, y \in V$ and any positive $\lambda \in F$,*

$$|\langle x, y \rangle + \langle y, x \rangle| \leq \lambda^{-1} \langle x, x \rangle + \lambda \langle y, y \rangle.$$

PROOF. If λ is any positive element of F , $0 \leq \langle x - \lambda y, x - \lambda y \rangle$. Expanding, rearranging, and dividing by λ gives

$$\langle x, y \rangle + \langle y, x \rangle \leq \lambda^{-1} \langle x, x \rangle + \lambda \langle y, y \rangle.$$

Similar treatment of $0 \leq \langle x + \lambda y, x + \lambda y \rangle$ then yields the result.

5.3. LEMMA. *Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be an infinite dimensional Hilbertian space satisfying OR. Let A be a subfield of F which is order isomorphic to a naturally ordered subfield of the real numbers and let \mathbf{R} denote an extension of A to the real numbers. If $\{a_i\}_{i=1}^\infty$ is a sequence from A with $a_1 \neq 0$ and which is square summable in \mathbf{R} to an element of A and if $\{u_i\}_{i=1}^\infty$ is an orthonormal sequence in V , then there is an $x = \sum_{i=1}^\infty b_i u_i \in V$ which satisfies:*

- (i) $(a_k - b_k)\theta(a_k - b_k)$ is infinitesimal for all k ;
- (ii) $\langle x, x \rangle - \sum_{i=1}^\infty a_i^2$ is infinitesimal.

PROOF. Let \mathcal{H} denote separable Hilbert space over \mathbf{R} and let $p = (a_i)$ and $q = (1, 0, 0, 0, \dots)$ in \mathcal{H} . Then set

$$p' = (p, p)(q, p)^{-1}q - p$$

and

$$q' = (p, p)(q, p)^{-1}q = ((p, p)(q, p)^{-1}, 0, 0, 0, \dots).$$

We have $q' = p + p'$ and $(p, p') = 0$. Applying 5.1 to p and p' in \mathcal{H} we obtain sequences $\{p_i\}$ and $\{p'_i\}$ from $\mathcal{H}^0 \cap \mathcal{H}(A)$ converging to p and p' , respectively. Now each element (λ_i) of $\mathcal{H}^0 \cap \mathcal{H}(A)$ corresponds to an element $\sum_{i=1}^\infty \lambda_i u_i$ of V and (\cdot, \cdot) acts on the first just as $\langle \cdot, \cdot \rangle$ does on the second. Let $\{r_i\}$ and $\{r'_i\}$ denote the sequences in V corresponding to the sequences $\{p_i\}$ and $\{p'_i\}$ in $\mathcal{H}^0 \cap \mathcal{H}(A)$. Now let ρ be a positive rational element of A and choose n large enough so that $\|p - p_n\| \leq \rho$ and $\|p' - p'_n\| \leq \rho$. Then $(q' - p_n - p'_n, q' - p_n - p'_n) \leq (2\rho)^2$ and with $z = (p, p)(q, p)^{-1}u_1$ we have $\langle z - r_n - r'_n, z - r_n - r'_n \rangle \leq (2\rho)^2$. Let M be the subspace of V generated by $\{r_i\}_{i=1}^\infty$. Then we may write $z = x + y$ with $x \in M^{\perp\perp}$ and $y \in M^\perp$. It follows that $\langle x - r_n, x - r_n \rangle \leq (2\rho)^2$. If M_1 is the subspace of V generated by $\{u_i\}_{i=1}^\infty$, then $M \subset M_1$ and, hence, $x \in M^{\perp\perp} \subset M_1^{\perp\perp}$. Thus $x = \sum_{i=1}^\infty b_i u_i$ for a proper choice of $b_i \in D$. We must show that x and p satisfy conditions (i) and (ii).

We begin with relation (i). Let k be a fixed integer. Let $p_n = (c_i)$. Then $r_n = \sum_{i=1}^\infty c_i u_i$. Now if s is the projection of $p - p_n$ onto the k th coordinate space then because $\|p - p_n\| \leq \rho$ we have $(s, s) \leq \rho^2$. Translating into V , s corresponds to $a_k u_k - c_k u_k$ yielding $\langle a_k u_k - c_k u_k, a_k u_k - c_k u_k \rangle \leq \rho^2$. Also the projection of $x - r_n$ onto the k th coordinate space is $b_k u_k - c_k u_k$ and by virtue of $\langle x - r_n, x - r_n \rangle \leq (2\rho)^2$ we have $\langle b_k u_k - c_k u_k, b_k u_k - c_k u_k \rangle \leq (2\rho)^2$. Now applying 5.2 we obtain

$$\begin{aligned}
& \langle a_k u_k - b_k u_k, a_k u_k - b_k u_k \rangle \\
&= \langle (a_k u_k - c_k u_k) + (c_k u_k - b_k u_k), (a_k u_k - c_k u_k) + (c_k u_k - b_k u_k) \rangle \\
&\leq \rho^2 + (2\rho)^2 + \langle a_k u_k - c_k u_k, c_k u_k - b_k u_k \rangle \\
&\quad + \langle c_k u_k - b_k u_k, a_k u_k - c_k u_k \rangle \\
&\leq 5\rho^2 + \rho^{-1}(\rho^2) + \rho(2\rho)^2 = \rho + 5\rho^2 + 4\rho^3.
\end{aligned}$$

Since by choosing n large enough ρ can be as small a rational as desired, (i) follows.

We move to the consideration of (ii). We must first establish an upper bound for $\langle r_n, r_n \rangle$. By a series of rearrangements and an application of 5.2,

$$\begin{aligned}
& \langle x - r_n, x - r_n \rangle \leq 4\rho^2, \\
& \langle x, x \rangle - \langle x, r_n \rangle - \langle r_n, x \rangle + \langle r_n, r_n \rangle \leq 4\rho^2, \\
& \langle r_n, r_n \rangle \leq 4\rho^2 - \langle x, x \rangle + (\langle r_n, x \rangle + \langle x, r_n \rangle) \\
& \leq 4\rho^2 - \langle x, x \rangle + 2\langle x, x \rangle + 2^{-1}\langle r_n, r_n \rangle, \\
& \langle r_n, r_n \rangle \leq 8\rho^2 + 2\langle x, x \rangle.
\end{aligned}$$

Employing this result and another application of 5.2 we have

$$\begin{aligned}
& \langle x - r_n, x - r_n \rangle \leq 4\rho^2, \\
& \langle x, x \rangle - \langle r_n, x \rangle - \langle x, r_n \rangle + \langle r_n, r_n \rangle \leq 4\rho^2, \\
& \langle x, x \rangle - \langle r_n, r_n \rangle \leq 4\rho^2 + \langle r_n, x \rangle + \langle x, r_n \rangle - 2\langle r_n, r_n \rangle, \\
& \langle x, x \rangle - \langle r_n, r_n \rangle \leq 4\rho^2 + \langle r_n, x - r_n \rangle + \langle x - r_n, r_n \rangle \\
& \leq 4\rho^2 + \rho^{-1}\langle x - r_n, x - r_n \rangle + \rho\langle r_n, r_n \rangle \\
& \leq 4\rho^2 + 4\rho + 8\rho^3 + 2\rho\langle x, x \rangle.
\end{aligned}$$

Starting with the same initial inequality and using similar rearrangements we may arrive at

$$\langle r_n, r_n \rangle - \langle x, x \rangle \leq 4\rho^2 + 4\rho + \rho\langle x, x \rangle.$$

Now $\langle x, x \rangle \leq \langle z, z \rangle$ and hence $\langle x, x \rangle$ is finite. It follows that for any preassigned rational $\varepsilon > 0$, for all n large enough, $|\langle x, x \rangle - \langle r_n, r_n \rangle| \leq \varepsilon$. By virtually identical arguments in \mathcal{H} we also have $|(p, p) - (p_n, p_n)| \leq \varepsilon$ for all n large enough. Thus for all n large enough $|\langle x, x \rangle - (p, p)| \leq 2\varepsilon$. Since this last expression does not depend on n the result (ii) follows.

While this lemma will prove useful in the non-Archimedean case, our first application will be a completeness result on Archimedean fields. Somewhat similar results in a different setting have been obtained by Morash [13] and Holland [9].

5.4. THEOREM. *Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be an infinite dimensional Hilbertian space satisfying OR. If there is an orthonormal sequence $\{u_i\}_{i=1}^\infty$ in V and if F is Archimedean, then F is isomorphic to the real numbers and $(V, \langle \cdot, \cdot \rangle, \theta, D)$ satisfies N.*

PROOF. It is only necessary to show that F is isomorphic to the real numbers because N is then an obvious consequence. Let $F = A$ in 5.3 and let $r \in \mathbb{R} - A$ where $r > 0$. Let q be a rational number with $0 < r < q$. Then a sequence of positive rationals $\{a_i\}_{i=1}^\infty$ may be found satisfying $\sum_{i=1}^\infty a_{2i}^2 = r$ and $\sum_{i=1}^\infty a_{2i-1}^2 = q - r$. Since infinitesimal means zero in F , 5.3 yields the existence of $x = \sum_{i=1}^\infty a_i u_i \in V$ with $\langle x, x \rangle = q$. Let M be the subspace of V generated by $\{u_{2i}\}_{i=1}^\infty$. Then we may write $x = s + t$, $s \in M^{\perp\perp}$ and $t \in M^\perp$. Clearly $s = \sum_{i=1}^\infty a_{2i} u_{2i}$ and $t = \sum_{i=1}^\infty a_{2i-1} u_{2i-1}$. Now if for each n , $s_n = \sum_{i=1}^n a_{2i} u_{2i}$ then

$$\sum_{i=1}^n a_{2i}^2 = \langle s_n, s_n \rangle \leq \langle s, s \rangle$$

for each n . It follows that $r \leq \langle s, s \rangle$. Likewise $q - r \leq \langle t, t \rangle$. Since $\langle s, s \rangle + \langle t, t \rangle = \langle x, x \rangle = q$, necessarily $r = \langle s, s \rangle$ and $r \in F$. This contradiction establishes the result.

We return now to a consideration of the probabilistic logics. We shall need several lemmas before we reach our main theorem.

5.5. LEMMA. *Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be an infinite dimensional Hilbertian space satisfying P_1 and P_2 . Let $\{u_i\}_{i=1}^\infty$ and $\{v_i\}_{i=1}^\infty$ be orthonormal sequences in V which are orthogonal to each other. Then if $x = \sum_{i=1}^\infty a_i u_i \in V$, we also have $x' = \sum_{i=1}^\infty a_i v_i \in V$ and $\langle x', x' \rangle = \langle x, x \rangle$.*

PROOF. Since 2^{-1} is positive in F there is by P_1 on $\alpha \in D$ with $\alpha\theta(\alpha) = 2^{-1}$. Now let M_1 and M_2 be the subspaces generated by the orthonormal sequences $\{\alpha u_i + \alpha v_i\}_{i=1}^\infty$ and $\{\alpha u_i - \alpha v_i\}_{i=1}^\infty$, respectively. Then we may write $x = r_1 + r_2$, $r_1 \in M_1^{\perp\perp}$ and $r_2 \in M_2^{\perp\perp}$. Since $\langle r_1, \alpha u_i + \alpha v_i \rangle = \langle x, \alpha u_i + \alpha v_i \rangle = a_i \theta(\alpha)$ and likewise $\langle r_2, \alpha u_i - \alpha v_i \rangle = a_i \theta(\alpha)$ we have

$$r_1 = \sum_{i=1}^\infty \frac{a_i}{2} u_i + \frac{a_i}{2} v_i, \quad r_2 = \sum_{i=1}^\infty \frac{a_i}{2} u_i - \frac{a_i}{2} v_i.$$

By a similar expansion of r_1 and r_2 in terms of $\{u_i\}_{i=1}^\infty$ and $\{v_i\}_{i=1}^\infty$, we may write $r_1 = w_1 + w_2$ and $r_2 = w_1 - w_2$, where $w_1 = \sum_{i=1}^\infty (a_i/2) u_i$ and $w_2 = \sum_{i=1}^\infty (a_i/2) v_i$. Clearly $\langle r_1, r_1 \rangle = \langle r_2, r_2 \rangle$. But $\langle r_1, r_1 \rangle + \langle r_2, r_2 \rangle = \langle x, x \rangle$. Thus $\langle r_1, r_1 \rangle = 2^{-1} \langle x, x \rangle$. Now $w_1 = 2^{-1} x$ so $\langle w_1, w_1 \rangle = 4^{-1} \langle x, x \rangle$. It follows that $\langle w_2, w_2 \rangle = 4^{-1} \langle x, x \rangle$. But $x' = 2w_2$, hence $x' \in V$ and $\langle x', x' \rangle = \langle x, x \rangle$.

5.6. LEMMA. *Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be an infinite dimensional Hilbertian*

space satisfying P_1 and P_2 . Let $\{u_i\}_{i=1}^\infty$ be an orthonormal sequence from V . Then if $x = \sum_{i=1}^\infty a_i u_i \in V$, we have $x' = \sum_{i=1}^\infty \theta(a_i) u_i \in V$ and $\langle x', x' \rangle = \langle x, x \rangle$.

PROOF. Without loss of generality we may assume $a_i \neq 0$ for all i and that $\{v_i\}_{i=1}^\infty$ is an orthonormal sequence in V which is orthonogonal to $\{u_i\}_{i=1}^\infty$. Let $\alpha \in D$ with $\alpha\theta(\alpha) = 2^{-1}$ as in the proof of 5.5. Then for each i define $\alpha_i = \theta(\alpha)$ and $\beta_i = \theta(\alpha) a_i^{-1} \theta(a_i)$. We have

$$\alpha_i \theta(\alpha_i) = \theta(\alpha) \alpha = 2^{-1}$$

and

$$\beta_i \theta(\beta_i) = \theta(\alpha) a_i^{-1} \theta(a_i) a_i \theta(a_i^{-1}) \alpha = 2^{-1}.$$

Thus the sequences $\{\alpha_i u_i + \beta_i v_i\}_{i=1}^\infty$ and $\{\alpha_i u_i - \beta_i v_i\}_{i=1}^\infty$ are mutually orthogonal and orthonormal and if they generate subspaces M_1 and M_2 of V we may write $x = r_1 + r_2$, $r_1 \in M_1^{\perp\perp}$ and $r_2 \in M_2^{\perp\perp}$. Here $\langle r_1, \alpha_i u_i + \beta_i v_i \rangle = \langle x, \alpha_i u_i + \beta_i v_i \rangle = a_i \alpha$ and we have

$$r_1 = \sum_{i=1}^\infty \frac{a_i}{2} u_i + \frac{\theta(a_i)}{2} v_i$$

with a similar relation for r_2 . Thus with $w_1 = \sum_{i=1}^\infty (a_i/2) u_i$ and $w_2 = \sum_{i=1}^\infty (\theta(a_i)/2) v_i$, we may argue just as in the proof of 5.5 to see that if $y = 2w_2$ then $\langle y, y \rangle = \langle x, x \rangle$. The remainder of the proof follows from 5.5.

5.7. LEMMA. Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be an infinite dimensional Hilbertian space satisfying P_1 and P_2 . Let $x = \sum_{i=1}^\infty a_i u_i \in V$ where $\{u_i\}_{i=1}^\infty$ is an orthonormal sequence from V . Then if $\{d_i\}_{i=1}^\infty$ is any sequence from D with $d_i \theta(d_i) < 4^{-1}$ for each i we have $y = \sum_{i=1}^\infty d_i a_i u_i \in V$ and $\langle y, y \rangle < \langle x, x \rangle$.

PROOF. Again let α be an element with $\alpha\theta(\alpha) = 2^{-1}$. Without loss of generality we may assume $\{v_i\}_{i=1}^\infty$ and $\{w_i\}_{i=1}^\infty$ are orthonormal sequences from V which are mutually orthogonal and orthogonal to $\{u_i\}_{i=1}^\infty$. Now for each i let $\alpha_i = \alpha$ and $\beta_i = \theta(\alpha^{-1}) \theta(d_i)$. Then

$$\begin{aligned} \beta_i \theta(\beta_i) &= \theta(\alpha^{-1}) \theta(d_i) d_i \alpha^{-1} = \theta(\alpha^{-1}) \alpha^{-1} \theta(d_i) d_i \\ &= (\alpha \theta(\alpha))^{-1} d_i \theta(d_i) < 2^{-1}. \end{aligned}$$

Since $\alpha_i \theta(\alpha_i) = 2^{-1}$ for each i it follows that for each i there is a γ_i with

$$\gamma_i \theta(\gamma_i) = 1 - \alpha_i \theta(\alpha_i) - \beta_i \theta(\beta_i).$$

Thus $\{\alpha_i u_i + \beta_i v_i + \gamma_i w_i\}_{i=1}^\infty$ is an orthonormal sequence in V . Let M be the subspace of V generated by this sequence and let $r = \sum_{i=1}^\infty \theta(a_i) u_i$. Then we may write $r = p + q$, $p \in M^{\perp\perp}$ and $q \in M^\perp$. A simple calculation of coefficients yields

$$p = \sum_{i=1}^{\infty} [(\theta(a_i)/2)u_i + \theta(a_i)\theta(d_i)v_i + \theta(a_i)\theta(\alpha)\gamma_i w_i].$$

If now M_1 is the subspace of V generated by $\{v_i\}_{i=1}^{\infty}$ we may write $p = s + t$, $s \in M_1^{\perp\perp}$ and $t \in M_1^{\perp}$. Here $s = \sum_{i=1}^{\infty} \theta(a_i)\theta(d_i)v_i$, and if we set

$$z = \sum_{i=1}^{\infty} \theta(\theta(a_i)\theta(d_i))v_i = \sum_{i=1}^{\infty} d_i a_i v_i,$$

we have by 5.5 and 5.6, $y \in V$ and

$$\langle y, y \rangle = \langle z, z \rangle = \langle s, s \rangle \leq \langle p, p \rangle \leq \langle r, r \rangle = \langle x, x \rangle.$$

We are now ready to prove our main theorem.

5.8. THEOREM. *Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be an infinite dimensional Hilbertian space satisfying P_1 and P_2 . Then one of the following three cases holds:*

- (i) $D = \mathbf{R}$, the real field, and $\theta = \text{identity}$;
- (ii) $D = \mathbf{R}(i)$, the complex field, and θ is the usual conjugation;
- (iii) $D = \mathbf{R}(i, j)$, the quaternions, and θ is the usual conjugation.

As a consequence, V with the form $\langle \cdot, \cdot \rangle$ is one of the classical Hilbert spaces over the reals, the complexes, or the quaternions.

PROOF. The crucial step here is to show that F is Archimedean, hence let us suppose it is not. By application of P_1 we may find an orthonormal sequence $\{u_i\}_{i=1}^{\infty}$ in V . Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of positive rational numbers in F such that $\sum_{i=1}^{\infty} a_i^2 = 3^{-1}$. Then applying 5.3 with A equal to the subfield of F consisting of the rationals we obtain an element $x = \sum_{i=1}^{\infty} b_i u_i \in V$ with $\langle x, x \rangle < 2^{-1}$ and such that $(a_i - b_i)\theta(a_i - b_i)$ is infinitesimal for each i . Knowing the action of θ and form of elements by 4.2 it is clear that each $b_i\theta(b_i)$ is a finite positive element of F . Now let f be some nonzero infinitesimal element from F . Then with $d'_i = fb_i^{-1}$ for each i we have

$$\begin{aligned} d'_i\theta(d'_i) &= fb_i^{-1}\theta(b_i^{-1})\theta(f) = f^2(\theta(b_i)b_i)^{-1} \\ &= f^2(b_i\theta(b_i))^{-1} < 4^{-1}. \end{aligned}$$

It follows by 5.7 that

$$y' = \sum_{i=1}^{\infty} f u_i \in V.$$

Thus

$$z = f^{-1}y' = \sum_{i=1}^{\infty} u_i \in V.$$

Now if z_n represents the projection of z onto the subspace generated by $\{u_i\}_{i=1}^n$ we have

$$n = \sum_{i=1}^n \langle u_i, u_i \rangle = \langle z_n, z_n \rangle \leq \langle z, z \rangle.$$

It follows that $\langle z, z \rangle = h$ is infinite. Now let $c \in D$ with $c\theta(c) = h^{-1}$. For each i let $d_i = cb_i^{-1}$. Then

$$d_i\theta(d_i) = cb_i^{-1}\theta(b_i^{-1})\theta(c) = c\theta(c)(b_i\theta(b_i))^{-1} \leq 4^{-1}$$

because h^{-1} is infinitesimal. By 5.7 with $y = \sum_{i=1}^{\infty} d_i b_i u_i$ we have $\langle y, y \rangle \leq \langle x, x \rangle < 2^{-1}$. However $y = cz$ and this demands $\langle y, y \rangle = 1$. From this contradiction we conclude that F is Archimedean.

The remainder of the proof is now simple. First from 5.4 it follows that $F = \mathbf{R}$, the real field. With this result (i)–(iii) are an almost trivial consequence of (i)–(iii) of 4.2. It is only necessary to note that in case (ii) of 4.2 if $F = \mathbf{R}$ then there is an $a \in F$ such that with $i = aI$, $i^2 = -1$ and likewise for case (iii). We conclude that V with the form $\langle \cdot, \cdot \rangle$ is a pre-Hilbert space over the reals, the complexes, or the quaternions. It then follows from a result due to Piron and to Araki [15, 7.42] that V is complete and thus a Hilbert space. This completes the proof of the theorem.

5.9. COROLLARY. *The only infinite dimensional complete projective logics which are probabilistic are the classical logics of the real, complex, or quaternionic Hilbert spaces.*

5.10. COROLLARY. *The only infinite dimensional complete projective logics over an algebraically closed field are the classical logics of the complex Hilbert spaces. Likewise the only infinite dimensional complete projective logics over a real closed field are the classical logics of the real Hilbert spaces.*

Here 5.10 is a consequence of the fact that in an algebraically closed or real closed field the condition N is satisfied regardless of the choice of the involutive antiautomorphism θ . This result is of interest in its relation to a theorem of Piron [14] (or see [15]) which asserts that if \mathcal{L} is a projective logic of dimension \aleph_0 over the complex field and if the corresponding involutive antiautomorphism θ is continuous (continuity need not be assumed in the real or quaternionic cases) then \mathcal{L} is just the logic of separable complex Hilbert space. Result 5.10 shows that in fact continuity need not be assumed in the complex case. Briefly, it is always possible to induce a new topology on the complex field using the original topology and a discontinuous field automorphism in such a way that θ is continuous. An example of an infinite dimensional logic over the complex field which illustrates these possibilities is given in [17, §7].

6. Axioms for the standard logic of separable complex Hilbert space. By employing the results of the previous section we are enabled to give here a

strictly lattice-theoretic characterization of the lattice of closed subspaces of separable complex Hilbert space. To begin we shall assume \mathcal{L} is a logic and we shall proceed by setting down a system of axioms which single out the standard logic of complex Hilbert space from the class of all logics.

We shall need a rather general definition of dimension in a lattice. Let S be a subset of \mathcal{L} and let x be an element in \mathcal{L} . If $t < x$ for each $t \in S$ we shall say x bounds S . Then following [2] we define the dimension of x or $d[x]$ to be the least cardinal which is an upper bound for all the cardinals of subsets S of \mathcal{L} which do not contain zero, are chains, and are bounded by x . We further set $d[\mathcal{L}] = d[1]$. According to this definition the points of \mathcal{L} are just the elements of dimension 1.

We first list the axioms and then discuss their sufficiency.

AXIOM I. \mathcal{L} has at least one point.

AXIOM II. $d[\mathcal{L}] = \aleph_0$.

AXIOM III. If a and a^\perp are nonzero elements of \mathcal{L} and x is a point in \mathcal{L} , then there are points y and z in \mathcal{L} with $y < a$, $z < a^\perp$ and $x < y \vee z$.

AXIOM IV. If a , b and c are elements of \mathcal{L} and if $a \vee b$ is less than the sum of a finite set of points in \mathcal{L} , and $c < a$, then $a \wedge (b \vee c) = (a \wedge b) \vee c$.

AXIOM V. If x and y are distinct points in \mathcal{L} then there is a distinct third point z in \mathcal{L} with $z < x \vee y$.

AXIOM VI. Given any four distinct points u, v, w, x with $u \vee v = w \vee x$ there exist points y and z in \mathcal{L} with $y < (v \vee z)$, $y < (w \vee x)$, $z < (w \vee x)$, $z < (u \vee y)$, and such that $d[w \vee x \vee y \vee z] = 2$ and

$$(u \vee y) \wedge (z \vee x) < [(v \vee z) \wedge (w \vee y)] \vee [(u \vee v) \wedge (z \vee y)].$$

AXIOM VII. If u, v, w, x, y and z are six distinct points in \mathcal{L} with $d[u \vee v \vee w \vee x \vee y \vee z] = 2$ and if $(u \vee v) \wedge (w \vee z) \wedge (y \vee z) \neq 0$ and $(x \vee y) \wedge (z \vee u) \wedge (v \vee w) \neq 0$, then $(u \vee x) \wedge (v \vee y) \wedge (w \vee z) \neq 0$.

Our first step in demonstrating the sufficiency of these axioms will be to show that they imply \mathcal{L} is a complete projective logic.

6.1. LEMMA. Any logic \mathcal{L} satisfying Axioms I–V is a complete projective logic.

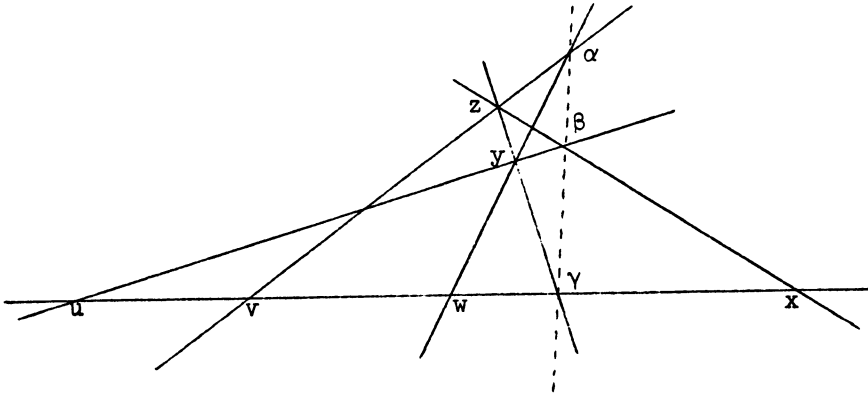
PROOF. The completeness follows from Axiom II and projectivity [15, p. 183] hence it is enough to establish that \mathcal{L} is projective. Let a be a nonzero element of \mathcal{L} and let x be a point of \mathcal{L} by Axiom I. If $a = 1$ then $x < a$. If $a \neq 1$ then $a^\perp \neq 0$ and by Axiom III and the existence of x there is a point less than a . Thus condition (i) of projectivity (see §2) holds. For condition (ii) let a be the sum of a nonempty finite set of points in \mathcal{L} . By the definition of a logic and by Axiom IV it is clear that $\mathcal{L}[0, a]$ is a complemented modular lattice of finite dimension. It is now a consequence of Axiom V and [15, 2.15] that $\mathcal{L}[0, a]$ is a geometry and condition (ii) is satisfied. Condition (iii) is just Axiom III while condition (iv) is a simple consequence of conditions (i) and

(ii), Axiom II and the definition of a logic.

It is now clear that if \mathcal{L} satisfies Axioms I–V then \mathcal{L} is represented by a Hilbertian space $(V, \langle \cdot, \cdot \rangle, \theta, D)$.

6.2. LEMMA. *The space $(V, \langle \cdot, \cdot \rangle, \theta, D)$ is a Hilbertian space representing a logic \mathcal{L} which satisfies Axiom VI if and only if every element of D has a square root in D .*

PROOF. Let $(V, \langle \cdot, \cdot \rangle, \theta, D)$ be a Hilbertian space representing a logic \mathcal{L} which satisfies Axiom VI. Now let u, v , and w be distinct points on a line c in \mathcal{L} and let a be some plane in \mathcal{L} containing c . Then we recall [15] that the multiplicative group $D' = D - \{0\}$ may be represented as the group $M(c; u, v)$ consisting of the identity and the general projectivities in $P_2(c)$ as determined in $\mathcal{L}[0, a]$ and with u and v as their first and second canonical fixed points, respectively. Further the elements of $M(c; u, v)$ are uniquely determined by their action on w and may be represented as $\{q_x: x < c, x \neq u \text{ and } x \neq v\}$ where $q_x(w) = x$. Thus we must establish that for any such x there is a γ with $q_\gamma \circ q_\gamma = q_x$. If $x = w$ we may set $\gamma = w$ for a solution so let us assume $x \neq w$. Then by Axiom VI there are points y and z in \mathcal{L} with prescribed properties in relation to the points u, v, w , and x . Let us assume for the moment that y and z are in $\mathcal{L}[0, a]$. Then with $\alpha = (v \vee z) \wedge (w \vee y)$, $\beta = (u \vee y) \wedge (z \vee x)$ and $\gamma = (u \vee v) \wedge (z \vee y)$, Axiom VI asserts that the figure is correct and α, β and γ are collinear.



It can then be easily seen from the figure using y and β as centers of perspectivity that $q_\gamma \circ q_\gamma = q_x$. Now if y and z are not in $\mathcal{L}[0, a]$ then there is some other plane b containing y, z , and c . But then by [15, 5.3] there is an isomorphism between $\mathcal{L}[0, a]$ and $\mathcal{L}[0, b]$ which leaves $\mathcal{L}[0, c]$ fixed and hence corresponding to y and z there are points y' and z' in $\mathcal{L}[0, a]$ which satisfy all the conditions of Axiom VI with respect to u, v, w , and x . Thus in any case

there is a q_y with $q_y \circ q_y = q_x$. This establishes the only if part of the lemma. The if part is trivial.

6.3. THEOREM. *A logic \mathcal{L} satisfies Axioms I–VII if and only if it is isomorphic to the standard logic of separable complex Hilbert space.*

PROOF. Suppose \mathcal{L} satisfies Axioms I–VII. Then by 6.1, \mathcal{L} may be represented by a Hilbertian space $(V, \langle \cdot, \cdot \rangle, \theta, D)$. By 6.2 every element of D has a square root in D and by Axiom VII, which is Pappus' theorem, D is a field. Thus $(V, \langle \cdot, \cdot \rangle, \theta, D)$ satisfies condition N and since \mathcal{L} is infinite dimensional it must by 5.9 be isomorphic to one of the classical logics of real, complex, or quaternionic Hilbert space. Since the reals do not have a square root for every element and the quaternions are not commutative we are left with the complex field. As Axiom II sets the dimension at \aleph_0 we conclude that \mathcal{L} is isomorphic to the logic of separable complex Hilbert space. The converse statement that the logic of separable complex Hilbert space satisfies Axioms I–VII follows easily from [15, 7.40], the result 6.2, and the fact that commutativity implies Pappus' theorem.

It is well known that one cannot hope to arrive at the complex field by purely algebraic hypotheses. In the case of the finite dimensional geometries, assumptions concerning the character of the topology which may be defined on the geometry have proved useful in singling out the geometries associated with the real, complex, or quaternionic numbers [10], [16], [18]. In our case, Axiom II and the countable completeness inherent in the definition of a logic supply the essential nonalgebraic character to arrive at the complex field.

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